Entanglement of graph states up to 8 qubits

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Abstract

The entanglement of graph states up to eight qubits is calculated in the regime of iteration calculation. The entanglement measures could be the relative entropy of entanglement, the logarithmic robustness or the geometric measure. All 146 local inequivalent graphs are classified as two categories: graphs with identical upper LOCC entanglement bound and lower bipartite entanglement bound, graphs with unequal bounds. The late may displays non-integer entanglement. The precision of iteration calculation of the entanglement is less than 10^{-14}.

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1 Introduction

Multipartite entanglement plays an important role in quantum error correction and quantum computation. The quantification of multipartite entanglement is still open even for a pure multipartite state. Until now, a variety of different entanglement measures have been proposed for multipartite setting. Among them are the *(Global) Robustness of Entanglement* [3] , the *Relative Entropy of Entanglement*[1] [2] , and *the Geometric Measure* [4]. The robustness measures the minimal noise (arbitrary state) that we need to added to make the state separable. The geometric measure is the distance of state to the closest product state in terms of the fidelity. The relative entropy of entanglement is a valid entanglement measure for multipartite state, it is the relative entropy of the state under consideration to the closest fully separable state.

The quantification of multipartite entanglement is usually very difficult as most measures are defined as the solutions to difficult variational problems. Even for pure multipartite state, the entanglement can only be obtained for some special scenario. Fortunately, due to the inequality on the logarithmic robustness, relative entropy of entanglement and geometric measure of entanglement[5] [6] [7], these entanglement measures are all equal for stabilizer states [8]. It is known that the set of graph states is a subset of the set of stabilizer states. Thus these entanglement measures are all equal for graph states.

The paper is organized as follows: In section 2, we overview the concept of graph state and graph state basis, the entanglement inequalities, the bounds of the entanglement. In section 3, we describe the iteration method of finding the closest product state. In section 4, we classify all the inequivalent graphs according to the entanglement bounds up to 8 qubits and propose the precise values of entanglement. Section 5 is devoted to the precision of the iteration. Conclusions are given in section 6.

2 Preliminary

2.1 Graph state

A graph $G = (V;\Gamma)$ is composed of a set V of n vertices and a set of edges specified by the adjacency matrix Γ, which is an $n\times n$ symmetric matrix with vanishing diagonal entries and $\Gamma_{ab} = 1$ if vertices a, b are connected and $\Gamma_{ab} = 0$ otherwise. The neighborhood of a vertex a is denoted by $N_a = \{v \in V \,|\Gamma_{av} = 1\}$, i.e, the set of all the vertices that are connected to a. Graph states [11] [12]] are useful multipartite entangled states that are essential resources for the one-way computing [13] and can be experimentally demonstrated [14]. To associate the graph state to the underlying graph, we assign each vertex with a qubit, each edge represents the interaction between the corresponding two qubits. More physically, the interaction may be Ising interaction of spin qubits. Let us denote the Pauli matrices at the qubit a by X_a, Y_a, Z_a and identity by I_a. The graph state related to graph G is defined as

$$|G\rangle = \prod_{\Gamma_{ab}=1} U_{ab} \,|+\rangle_x^V = \frac{1}{\sqrt{2^n}} \sum_{\mu=\mathbf{0}}^{\mathbf{1}} (-1)^{\frac{1}{2}\mu\Gamma\mu^T} \,|\mu\rangle_z \qquad (1)$$

where $|\mu\rangle_z$ is the joint eigenstate of Pauli operators Z_a ($a \in V$) with eigenvalues $(-1)^{\mu_a}$, $|+\rangle_x^V$ is the joint $+1$ eigenstate of Pauli operators X_a ($a \in V$) , and U_{ab} ($U_{ab} = diag\{1, 1, 1, -1\}$ in the Z basis) is the controlled phase gate between qubits a and b. Graph state can also be viewed as the result of successively performing 2-qubit Control-Z operations U_{ab} to the initially unconnected n qubit state $|+\rangle_x^V$. It can be shown that graph state is the joint $+1$ eigenstate of the n vertices stabilizers

$$K_a = X_a \prod_{b\in N_a} Z_b := X_a Z_{N_a},\ a \in V. \qquad (2)$$

Meanwhile, the graph state basis are $|G_{k_1,k_2,\cdots k_n}\rangle = \prod_{a\in V} Z_a^{k_a} |G\rangle$, with $k_a = 0, 1$. Thus

$$K_a \,|G_{k_1,k_2,\cdots k_n}\rangle = (-1)^{k_a} \,|G_{k_1,k_2,\cdots k_n}\rangle \,. \qquad (3)$$

Since all of the graph basis states are local unitary equivalent, they all have equal entanglement, so we only need

to determine the entanglement of graph state $|G\rangle$. Once the entanglement of a graph state is obtained, the entanglement of all the graph basis states are obtained.

A widely used local operation in dealing with graph states is the so-called local complementation (LC)[15], which is a multi-local unitary operation V_a on the $a-th$ qubit and its neighbors, defined as $V_a = \sqrt{K_a} = \exp(-i\frac{\pi}{4}X_a)\prod_{b\in N_a}\exp(i\frac{\pi}{4}Z_b)$. LC centered on a qubit a is visualized readily as a transformation of the subgraph of a-th qubit's neighbours, such that an edge between two neighbours of a is deleted if the two neighbours are themselves connected, or an edge is added otherwise.

2.2 Entanglement inequalities

The global robustness of entanglement [3] is defined as

$$R(\rho) = \min_{\omega} t \tag{4}$$

such that there exists a state ω such that $(\rho + t\omega)/(1+t)$ is separable. The logarithmic robustness [9] is

$$LR(\rho) = \log_2(1 + R(\rho)). \tag{5}$$

The relative entropy of entanglement is defined as the "distance" to the closest separable state with respect to the relative entropy [2],

$$E_r(\rho) = \min_{\omega\in Sep} S(\rho\|\,\omega), \tag{6}$$

where $S(\rho\|\,\omega) = -S(\rho) - tr\{\rho log_2\omega\}$ is the relative entropy, $S(\rho)$ is the von Neumann entropy, and Sep is the set of separable states. The minimum is taken over all fully separable mixed states ω.

The geometric measure of entanglement for pure state $|\psi\rangle$, is defined as

$$E_g(|\psi\rangle) = \min_{|\phi\rangle\in Pro} -\log_2 |\langle\phi|\,\psi\rangle|^2, \tag{7}$$

where Pro is the set of product states.

Hayashi *et al* [6] has been shown that the maximal number N of pure states in the set $\{|\psi_i\rangle\,|i = 1, ..., N\}$, that can be discriminated perfectly by LOCC is bounded by the amount of entanglement they contain:

$$\log_2 N \le n - \overline{LR(|\psi_i\rangle)} \le n - \overline{E_r(|\psi_i\rangle)} \le n - \overline{E_g(|\psi_i\rangle)}, \tag{8}$$

where $n = \log_2 D_H$, D_H is the total dimension of the Hilbert space, and $\overline{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$ denotes the "average".

For stabilizer state $|S\rangle$, it has been shown that[8]

$$LR(|S\rangle) = E_r(|S\rangle) = E_g(|S\rangle)$$

The technique is to prove $LR(|S\rangle) = E_g(|S\rangle)$ and utilizing inequality (8).The entanglement can be written as

$$E(|S\rangle) = \min_{\phi} -\log_2 |\langle S|\,\phi\rangle|^2, \tag{9}$$

where $|\phi\rangle = \prod_j(\sqrt{p_j}\,|0\rangle + \sqrt{1-p_j}e^{i\varphi_j}\,|1\rangle)$ is the product pure state.

2.3 Entanglement bounds

The entanglement is upper bounded by the local operation and classical communication (LOCC) bound $E_{LOCC} = n - \log_2 N$, and lower bounded by some bipartite entanglement deduced from the state, that is, the 'matching' bound E_{bi} [10]. It is well known that all graph states are stabilizer states, so the inequality for the entanglement of a graph state is

$$E_{bi} \le E \le E_{LOCC}. \tag{10}$$

If the lower bound coincides with the upper bound, the entanglement of the graph state can be obtained. This is the case for '2-colorable' graph states such as multipartite GHZ states, Steane code, cluster state, and state of ring graph with even vertices. For a state of ring graph with odd n vertices, we have $\lfloor\frac{n}{2}\rfloor \le E \le \lceil\frac{n}{2}\rceil$ [10].

The fidelity $F_\phi = |\langle G|\,\phi\rangle|^2$plays a crucial rule in calculating the entanglement. For a graph state, we have

$$E = \min_{\phi\in Pro} -\log_2 |\langle G|\,\phi\rangle|^2 = -\log_2(\max_{\phi\in Pro} F_\phi). \tag{11}$$

Denote $F = \max_{\phi\in Pro} F_\phi$ as the fidelity between the graph state and the closest pure separable state. The upper LOCC bound for a graph state is

$$E \le n - |A|, \tag{12}$$

since the largest number of entanglement basis states is $2^{|A|}$,where $|A|$ is the largest number of vertices with any two of the vertices being not adjacent [10].

The entanglement is lower bounded by the entanglement of a bipartition of the graph. The lower bound can be found by "matching"[10]. A convenient way of finding the lower bound of the entanglement is to find the largest set of non-adjacent edges first, then assign the two vertices of each edge to two parties to form a bipartition of the graph. The vertices that are not assigned can be assigned to either parties. To verify if these edges are the last Bell pairs that can be obtained, one can apply local Control-Z and LC to delete the redundant adjacent edges. It can be verified that all the graphs up to 8 qubits in the literatures can be treated in this manner to obtain the lower bound of the entanglement. This is not a difficult task since most of the graphs in the literatures are already in the simplest LC equivalent form. Thus, at least for graph states up to 8 qubits, the lower bound of the entanglement can be obtained by counting the largest number of non-adjacent edges.

3 Iterative method for the closest product states

If the upper LOCC bound coincides with the lower bipartition bound, the entanglement of the graph state can be determined and equals to the bounds. Still there are graph states that the two bounds do not meet. We need a systematical method to calculate the entanglement of such

graph states according to Eq. (11). The product pure state $|\phi\rangle$ can be denoted as

$$|\phi\rangle = \prod_j (x_j |0\rangle + y_j |1\rangle) \tag{13}$$

where x_j and y_j are complex numbers subjected to $|x_j|^2 + |y_j|^2 = 1$. Denote $f = \langle G| \phi\rangle$, then

$$f = \frac{1}{\sqrt{2^n}} \sum_{\mu=\mathbf{0}}^{\mathbf{1}} (-1)^{\frac{1}{2}\mu\mathbf{\Gamma}\mu^T} \prod_j (x_j^{1-\mu_j} y_j^{\mu_j}). \tag{14}$$

Let $L = |f|^2 - \sum_j \lambda_j (|x_j|^2 + |y_j|^2 - 1)$, where λ_j are the Lagrange multipliers. Then $\frac{\partial L}{\partial x_j} = 0$ and $\frac{\partial L}{\partial y_j} = 0$ lead to

$$\frac{\partial f}{\partial x_j} f^* - \lambda_j x_j^* = 0, \tag{15}$$

$$\frac{\partial f}{\partial y_j} f^* - \lambda_j y_j^* = 0. \tag{16}$$

The two equations are combined to

$$y_j^* \frac{\partial f}{\partial x_j} - x_j^* \frac{\partial f}{\partial y_j} = 0. \tag{17}$$

The left hand of Eq. (17) is $\langle G| \phi_j\rangle$, with

$$\begin{aligned} |\phi_j\rangle &= \prod_{k=1}^{j-1} (x_k |0\rangle + y_k |1\rangle)(y_j^* |0\rangle - x_j^* |1\rangle) \\ &\times \prod_{m=j+1}^{n} (x_m |0\rangle + y_m |1\rangle). \end{aligned} \tag{18}$$

Thus Eq. (4) is to say that the graph state is orthogonal to all $|\phi_j\rangle$ $(j = 1, \ldots, n)$ when $|\phi\rangle$ is the closest product state. It is clear that $|\phi\rangle$ is orthogonal to all $|\phi_j\rangle$ $(j = 1, \ldots, n)$ too. Denote $z_j = y_j/x_j$,the derivatives are

$$\begin{aligned} \frac{\partial f}{\partial x_j} &= \frac{1}{\sqrt{2^n}} \sum_{\nu=\mathbf{0}}^{\mathbf{1}'} (-1)^{\frac{1}{2}\nu\mathbf{\Gamma}\nu^T} \prod_{k\neq j} (x_k^{1-\mu_k} y_k^{\mu_k}) \\ &= \frac{\prod_{k\neq j} x_j}{\sqrt{2^n}} \sum_{\nu=\mathbf{0}}^{\mathbf{1}'} (-1)^{\frac{1}{2}\nu\mathbf{\Gamma}\nu^T} \prod_{k\neq j} z_k^{\mu_k}, \end{aligned} \tag{19}$$

$$\begin{aligned} \frac{\partial f}{\partial y_j} &= \frac{1}{\sqrt{2^n}} \sum_{\nu'=\mathbf{0}'}^{\mathbf{1}} (-1)^{\frac{1}{2}\nu'\mathbf{\Gamma}\nu'^T} \prod_{k\neq j} (x_k^{1-\mu_k} y_k^{\mu_k}) \\ &= \frac{\prod_{k\neq j} x_j}{\sqrt{2^n}} \sum_{\nu'=\mathbf{0}'}^{\mathbf{1}} (-1)^{\frac{1}{2}\nu'\mathbf{\Gamma}\nu'^T} \prod_{k\neq j} z_k^{\mu_k}. \end{aligned} \tag{20}$$

where $\nu = \{\mu_1, \ldots, \mu_{j-1}, 0, \mu_{j+1}, \ldots, \mu_n\}$, $\nu' = \{\mu_1, \ldots, \mu_{j-1}, 1, \mu_{j+1}, \ldots, \mu_n\}$, and $\mathbf{1}' = \{1, \ldots, 1, 0, 1, \ldots, 1\}$, $\mathbf{0}' = \{0, \ldots, 0, 1, 0, \ldots, 0\}$. The binary vector $\mathbf{1}'$ has all its entries being 1 except the $j-th$ entry being 0. $\mathbf{0}'$ is the logical NOT of $\mathbf{1}'$. From Eq. (17) we obtain the iterative equations for z_j,

$$z_j^* = \frac{\sum_{\nu'=\mathbf{0}'}^{\mathbf{1}} (-1)^{\frac{1}{2}\nu'\mathbf{\Gamma}\nu'^T} \prod_{k\neq j} z_k^{\mu_k}}{\sum_{\nu=\mathbf{0}}^{\mathbf{1}'} (-1)^{\frac{1}{2}\nu\mathbf{\Gamma}\nu^T} \prod_{k\neq j} z_k^{\mu_k}}. \tag{21}$$

We consider the change of fidelity in one step of iteration, that is, we only renew z_j according to Eq. (21) while keeping all the other z_k $(k \neq j)$ invariant in the step. Let $h_j = \frac{\partial f}{\partial x_j}$, $g_j = \frac{\partial f}{\partial y_j}$, then h_j and g_j are invariant in the step, $f = x_j h_j + y_j g_j$. Forget the iteration equation for a while, we seek the maximization of the fidelity with respect to $x_j = \cos\theta, y_j = \sin\theta e^{i\varphi}$. The maximal fidelity should be $|f|^2 = |h_j|^2 + |g_j|^2$,which is achieved when

$$z_j = \frac{y_j}{x_j} = \frac{g_j^*}{h_j^*}. \tag{22}$$

The condition (22) of maximal fidelity is just the iterative equation (21). Thus in each step of the iteration, the fidelity does not decrease. The fidelity increases abruptly or keeps unchanged in one step. In fact in each step, the fidelity can increase continuously from its initial value to its final value by changing (θ, φ) continuously.

Starting with any initial complex random vector $\mathbf{z} = (z_1, \ldots, z_n)$, the iterative equation renews each z_j successively, the fidelity increases (or does not change). After all z_j are renewed, a new round of iteration starts. The whole picture of iteration can be seen as a discrete process of the fidelity. The fidelity increases in each step until it does not increase any more. There may be the case that the fidelity reaches its local maximum. To find the global maximum, we run the iterative algorithm many times with random initial $\mathbf{z}$. Moreover, we calculate the fidelity of the graph state with respect to random separate states for a million times to determine roughly the possible range of the fidelity before the iteration calculation.

4 Classification of the graph state up to 8 qubits

The LC inequivalent graphs up to 7 qubits are all plotted in [11] and numbered. There are a two qubit graph (No.1), a three qubit graph (No.2), 2 of four qubit graphs (No.3 and No.4), 4 of five qubit graphs (No.5 to No.8), 11 of six qubit graphs (No.9 to No.19), 26 of seven qubit graphs (No.20 to No.45). In [16], the authors plotted all 101 LC inequivalent graphs of 8 qubits. The graphs of 8 qubits are numbered from No.46 to No.146.

4.1 Graph states with equal lower and upper bounds

The entanglement of graph states with equal lower and upper bounds can be calculated with the methods in Ref. [10]. It is listed in Table 1 and Table 2 for completeness.

The graphs that are "2-colorable" up to 8 qubits are listed in Table 1. It is a well known fact in graph theory that a graph is 2-colorable iff it does not contain any cycles of odd length. The LOCC upper bound and the lower bipartite bound of the entanglement for "2-colorable" graph state can be obtained by the methods described in Ref. [10]. For each of these "2-colorable" graph states, it has

Table 1

E	No.
1	1,2,3,5,9,20,46
2	4,6,7,10-12,15,21-24,31,47-51,69-70
3	13-14,18,25-30,34,38,43,52-63,74,76-77,81,
3	83-84,103-104,122
4	64-68,87,89,91,95,99-100,120,128,143

Table 2

E	No.
3	16-17,32-33,35-37,71-73,75,78-80,82,102,121
4	86,88,90,92-94,96-98,106-119,123-127,129-132,
4	135,136*,144

*the LC equivalent of No.136

been found that the two bounds coincide with each other $E_{LOCC} = E_{bi} = E_r$, and the relative entropy of entanglement is equal to the entanglement in Schmidt measure [11], $E_r = E_S = E$. Table 1 shows the results.

The LOCC bound for a "non 2-colorable" graph can be obtained with the largest set of non-adjacent vertices. The lower bipartition bound can be found by first searching for the largest set of non-adjacent edges, then verifying the candidate Bell pairs with local Control-Z and LC. Thus for "non 2-colorable" graph, the entanglement bounds of graph state can be obtained with "balls" (vertices) and "sticks" (edges) in the graph. When $E_{LOCC} = E_{bi} = E_r$, the graph states are shown in Table 2 (with $E = E_r = E_S$). No.101 graph is special for the graph state has the relative entropy of entanglement $E_r = 4$, the Schmidt measure $E_S = 3$.

4.2 Graph states with unequal bounds

Up to 8 qubits, what left are the "non 2-colorable" graph states whose upper entanglement bound E_{LOCC} (E_u) and lower bound E_{bi} (E_l) do not coincide. We utilize Eq. (21) to iteratively calculate the entanglement and find the closest product state with random initial complex numbers for z_j $(j = 1, \ldots, n)$. The values of relative entropy of entanglement are listed in Table 3.

A detail comparison of computed closest product states of No.8, No.39, No.41, No.45, No.85, No.105, No.134,No.137,No.138, No.140 shows that all these closest states have a substructure of the closest product state of ring 5 graph (No.8), although graph No.140 does not contain ring 5 graph explicitly, graph No.45 seems to contain graph No.19 as its subgraph. Ring 5 graph is essential to all these graph states with entanglement $k + 0.9275$ (integer k). In Ref. [17] an identical product closest state is supposed for ring 5 graph state, and it has been shown that the entanglement of ring 5 graph state is

$$E_{ring5} = 1 + \log_2 3 + \log_2(3 - \sqrt{3}) \approx 2.9275. \quad (23)$$

Denote $|\Phi_j\rangle = \sqrt{p}\,|0\rangle + \sqrt{1-p}e^{i\varphi_j}\,|1\rangle$,$(j = 1, \ldots, 4)$, with $\sqrt{p} = \sqrt{\frac{1}{2}(1 - \frac{1}{\sqrt{3}})} \approx 0.4597$, $\varphi_1 = \frac{\pi}{4}, \varphi_2 = -\frac{\pi}{4}, \varphi_3 = \frac{3\pi}{4}, \varphi_4 = -\frac{3\pi}{4}$. Typically, the closest product state of ring 5 graph state is

$$|\phi_{ring5}\rangle = |\Phi_1\rangle^{\otimes 5}, \quad (24)$$

The other graph states may have their closest product states

$$\begin{aligned}
|\phi_{No.39}\rangle &= |-\rangle\,|0\rangle\,|\Phi_3\rangle\,|\Phi_2\rangle^{\otimes 4}, & (25)\\
|\phi_{No.41}\rangle &= |\Phi_4\rangle\,|\Phi_1\rangle^{\otimes 3}\,|\Phi_4\rangle\,|0\rangle\,|-\rangle, & (26)\\
|\phi_{No.45}\rangle &= |-\rangle)\,|0\rangle\,|\Phi_4\rangle\,|\Phi_3\rangle^{\otimes 3}\,|\Phi_4\rangle, & (27)\\
|\phi_{No.85}\rangle &= |-\rangle^{\otimes 2}\,|\Phi_1\rangle^{\otimes 4}\,|\Phi_4\rangle\,|0\rangle, & (28)\\
|\phi_{No.105}\rangle &= |+\rangle^{\otimes 2}\,|\Phi_1\rangle^{\otimes 5}\,|1\rangle, & (29)\\
|\phi_{No.134}\rangle &= |-\rangle^{\otimes 2}\,|\Phi_4\rangle\,|\Phi_3\rangle^{\otimes 3}\,|\Phi_4\rangle\,|0\rangle, & (30)\\
|\phi_{No.137}\rangle &= |+\rangle\,|0\rangle\,|\Phi_4\rangle\,|\Phi_1\rangle^{\otimes 3}\,|\Phi_4\rangle\,|0\rangle, & (31)\\
|\phi_{No.138}\rangle &= |\Phi_2\rangle\,|\Phi_3\rangle^{\otimes 2}\,|0\rangle\,|+\rangle\,|0\rangle\,|\Phi_3\rangle^{\otimes 2}, & (32)\\
|\phi_{No.140}\rangle &= |1\rangle\,|\Phi_2\rangle^{\otimes 2}\,|\Phi_4\rangle^{\otimes 3}\,|0\rangle\,|-\rangle, & (33)
\end{aligned}$$

where $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$.

The next graph set (No.19, No.139, No.141) with non-integer entanglement ($k+0.5850$) graph states is specified by No.19 ([[6,0,4]] stabilizer state). The entanglement of No.19 is

$$E_{No.19} = 2 + \log_2 3 \approx 3.5850. \quad (34)$$

Typically, the closest product state is

$$|\phi_{No.19}\rangle = |\Phi_3\rangle^{\otimes 3}\,|\Phi_4\rangle^{\otimes 3}, \quad (35)$$

The closest states for No.139 and No.141 graph state can be

$$\begin{aligned}
|\phi_{No.139}\rangle &= |-\rangle\,|1\rangle\,|\Phi_4\rangle\,|\Phi_3\rangle^{\otimes 3}\,|\Phi_4\rangle\,|\Phi_1\rangle. & (36)\\
|\phi_{No.141}\rangle &= |-\rangle\,|0\rangle\,|\Phi_2\rangle\,|\Phi_4\rangle^{\otimes 3}\,|\Phi_3\rangle\,|\Phi_2\rangle. & (37)
\end{aligned}$$

No.139 and No.141 graphs have No.19 as their subgraph.

The entanglement of No.133 graph state is

$$\begin{aligned}
E_{No.133} &= 2 + 3\log_2 3 + \log_2(2 - \sqrt{3}) & (38)\\
&\approx 4.8549, & (39)
\end{aligned}$$

its closest product state can be

$$|\phi_{No.133}\rangle = |\Phi_4\rangle\,|\Phi_1\rangle^{\otimes 2}\,|\Phi_4\rangle^{\otimes 2}\,|\Phi_1\rangle^{\otimes 2}\,|\Phi_4\rangle. \quad (40)$$

The entanglement of No.44 is $E_{No.44} = 4$, its closest product state can be

$$|\phi_{No.44}\rangle = |\bigcirc\rangle^{\otimes 2}\,|+\rangle^{\otimes 3}\,|\bigcirc\rangle\,|-\rangle. \quad (41)$$

where $|\bigcirc\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i\,|1\rangle)$. The closest product states for No.40,42,142,145,146 can also be obtained, the iteration calculation should be modified as explained in the next section.

Table 3

No.	E_u	E_l	E_r	E_S	P_s
8	3	2	2.9275	2-3	0.997
19	4	3	3.5850	3-4	1.000
39	4	3	3.9275	3-4	0.790
40	4	3	4	3-4	0.241(1)
41	4	3	3.9275	3-4	0.264
42	4	3	4	3-4	0.950(1)
44	4	3	4	3-4	0.432
45	4	3	3.9275	3-4	0.967
85	4	3	3.9275	3-4	0.689
105	4	3	3.9275	3-4	0.658
133	5	4	4.8549	4-5	0.969
134	4	3	3.9275	3	0.646
137	5	4	4.9275	4-5	0.917
138	5	4	4.9275	4-5	0.617
139	5	4	4.5850	4-5	0.999
140	5	4	4.9275	4-5	0.571
141	5	4	4.5850	4	0.935
142	5	4	5	4-5	0.281(1)
145	5	4	5	4-5	0.870(2)
146	5	4	5	4-5	0.501(1)

$E_u = E_{LOCC}, E_l = E_{bi}$

5 Precision of iteration

We concentrate on the precision of iteration for calculating the entanglement of graph state whose lower and upper bounds do not meet. Let $\Delta = |E_{numeric} - E_{theory}|$ be the computational error of the iteration, where $E_{numeric}$ is the entanglement determined by iteration, E_{theory} $(= E_r)$ is the entanglement proposed in the former section . We use the exact value of E_{theory} rather than its approximation. For simplicity, we just give the successful probabilities of achieving the precision within $\Delta \leq 10^{-14}$ for some reasonable rounds of iteration with random initial conditions. From the actual numerical calculations, we can see that a precision of 10^{-14} is limited by the computer for our iterative algorithm (without double precision calculation).

For all graph states presented in Table 3 except No.40, 42, 142, 145, 146, the algorithm can be applied directly. The successful probabilities (P_s) are listed in Table 3. The round of the iteration is set to 150 except for No.140, whose round of iteration is 300. We renew $\mathbf{z}$ after each round instead of renewing z_j after each step of the round in the actual calculation for the reason of programming. In order to calculate the successful probability, we run the algorithm 1000 times for each graph state to count the number of algorithm that achieves the precision within $\Delta \leq 10^{-14}$.

For No.40, 42, 142, 145, 146 graph states, direct application of iterative algorithm fails. The numerical results of entanglement are all greater than the values given in Table 3, but the precision of the calculation is far from satisfactory. The precision can be 10^{-4} or so. A detail analysis of the separable state which gives best numerical value of entanglement shows us that the iterative equations (21) are correlated. The common figure of these nonlinear correlations of equations can be illustrated by applying iterative algorithm to the simplest graph state, the No.1 graph state (Bell pair). The iterative equations should be

$$z_1^* = \frac{1-z_2}{1+z_2}, \tag{42}$$

$$z_2^* = \frac{1-z_1}{1+z_1}. \tag{43}$$

Substituting Eq.(43) into Eq.(42), we obtain the identity $z_1^* = z_1^*$. Thus the two equations are correlated. The correlation of equations leads to the fail of iteration. We can delete one of Eq.(42) into Eq.(43) to solve the problem. The fidelity is

$$|f|^2 = \frac{|1+z_1+z_2-z_1z_2|^2}{4(1+|z_1|^2)(1+|z_1|^2)} \tag{44}$$

Applying Eq.(42) and ignoring Eq.(43), we obtain the correct maximal fidelity $|f|^2 = \frac{1}{2}$. Thus to obtain the maximal fidelity, we should omit some of the equations and use the remain equations for iteration. For No.40, 42,142 and 146, we omit one of the equations, indicated in Table 3 with notation (1) behind the successful probabilities. For No.145, we omit two of the equations, indicated in Table 3 with notation (2) behind the successful probability. In the numerical calculations, we set one or two z_i to random numbers that do not change in the iteration, respectively. Since we do not know if all the equations are correlated or only some of them are correlated, we calculate all possible choices of fixing z_i. For a given graph state, some of the choices of fixing z_i may not lead to sufficiently high successful probabilities or simply fail. The successful probabilities shown in Table 3 are the best.

We can see that the entanglement of all graph states in Table 3 can be efficiently calculated by iterative algorithm with very high precision. Most of them can be calculated directly, five of them can be calculated with modified iterative algorithm.

A heuristic point of view is that we can set the fixed z_i to be 0. For an n vertices graph state $|G_n\rangle$, the closest separable state should be $|\phi_n\rangle = |\phi_{n-1}\rangle |0\rangle$ when we set $z_n = 0$ without loss of generality. Denote the n bit binary vector μ_n as (μ_{n-1}, μ_n), and the $n \times n$ adjacent matrix Γ_n as $\begin{pmatrix} \Gamma_{n-1} & c^T \\ c & 0 \end{pmatrix}$,then

$$\frac{1}{2}(\mu_{n-1},0)\Gamma_n(\mu_{n-1},0)^T = \frac{1}{2}\mu_{n-1}\Gamma_{n-1}\mu_{n-1}^T. \tag{45}$$

Since $\langle\phi_n\,|\mu_n\rangle = \langle\phi_{n-1}|\langle 0|\,\mu_n\rangle = \langle\phi_{n-1}|\,\mu_{n-1}\rangle\,\delta_{0\mu_n}$, from the definition of graph state, we have

$$\langle G_n\,|\phi_{n-1}\rangle\,|0\rangle = \frac{1}{\sqrt{2}}\,\langle G_{n-1}|\,\phi_{n-1}\rangle\,. \tag{46}$$

Where G_{n-1} is the subgraph of G_n. From Eq.(46), a general relation for entanglement of any graph state and its

subgraph state follows

$$E_n \leq E_{n-1} + 1. \tag{47}$$

The equality holds for the case when $|\phi_n\rangle = |\phi_{n-1}\rangle|0\rangle$ is the closest separable state. For graph states No.40, 42,142,145,146, we can choose $|\phi_n\rangle = |\phi_{n-1}\rangle|0\rangle$ as the closest separable state when $|\phi_{n-1}\rangle$ is the closest separable state of the subgraph state, the calculation of the entanglement can be reduced, we have

$$E_n = E_{n-1} + 1. \tag{48}$$

for these graph states. The subgraphs of No.40 and No.42 belong to the set $\{No.13, No.14, No.17, No.18\}$,the entanglement of all the subgraph states is 3. Thus the entanglement of No.40 and No.42 is 4. The subgraphs of No.142, No.145 and No.146 belong to the set $\{No.40, No.42, No.44\}$,the entanglement of all the subgraph states is 4. Thus the entanglement of No.142, No.145 and No.146 is 5.

6 Conclusions

The entanglement of graph states measured in terms of the relative entropy of entanglement (also with logarithmic robust, the geometric measure) is given up to eight qubits. We use the iterative method to calculate the fidelity of graph state with respect to closest separable pure state. The iterative equations are results of the maximization of the fidelity. The equations have a clear meaning that graph state should be orthogonal to all other separable states that are orthogonal to the closest separable state. We have proved that in each step of a round of iteration the fidelity does not decrease. The iteration calculation has a very high efficiency if the resultant closest state does not contain $|0\rangle$ or $|1\rangle$ at all qubits (No.8, No.19, No.133). The precision of the iteration calculation can be less than 10^{-14} for all graph states up to 8 qubits with unequal lower and upper bounds of entanglement. To avoid possible missing of the global maximum, with random initial parameters, we calculate the fidelity for each graph state a million times without iteration to determine its rough range, and calculate the maximal fidelity 1000 times with iteration. Iterative method brings us with the exact entanglement value of the graph state if we substitute the numerical closest separable state with its nearest exact one. For a given graph state, there are many local equivalent closet separable states, they all lead to the same exact value of the entanglement. The precision of the numerical calculation is defined as the difference of the numerical and the exact entanglement. For some of the graph states, the iterative equations may correlate with each other. We analyze the situations and present a revised iterative algorithm to obtain the entanglement. In all the cases of unequal bounds, the entanglement may be equal to its upper bound (integer) or in between the bounds (not to be an integer). For all non-integer entanglement cases discussed, we have found that the qubit states $|\Phi_i\rangle$ are the indispensable ingredients of the closest separable states. Based on our calculation, the non-integer entanglement graphs could be further classified according to the number of $|\Phi_i\rangle$ in the closest separable state.

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References

[1] V. Vedral, M. B. Plenio, M. A. Rippin, P. L. Knight, Phys.Rev.Lett. **78,** 2275(1997) .

[2] V. Vedral and M. B. Plenio, Phys. Rev. A **57**, 1619(1998) .

[3] G. Vidal and R. Tarrach, Phys. Rev. A **59**, 141(1999).

[4] T-C. Wei, P. M. Goldbart, Phys.Rev.A, **68**, 042307(2003).

[5] T-C.Wei, M. Ericsson, P. M. Goldbart andW. J. Munro. Quant. Inform. Comp. **4**, 252(2004).

[6] M. Hayashi, D. Markham, M. Murao, M. Owari and S.Virmani, Phys. Rev. Lett. **96**, 040501(2006).

[7] T-C. Wei, Phys. Rev. A, **78**, 012327(2008) .

[8] M. Hayashi, D. Markham, M. Murao, M. Owari and S.Virmani, Phys. Rev. A, **77**, 012104 (2008).

[9] D. Cavalcanti, Phys. Rev. A 73,(2006) 044302 .

[10] D. Markham, A. Miyake, and S. Virmani, New. J. Phys. **9**, 194(2007).

[11] M. Hein, J. Eisert, and H. J. Briegel, Phys. Rev. A **69**,062311(2004).

[12] D. Schlingemann and R. F. Werner, Phys. Rev. A **65**, 012308(2002).

[13] R. Raussendorf and H. J. Briegel, Phys. Rev. Lett. **86**, 5188(2001).

[14] P.Walther, et.al., Nature **434**,(2005) 169 ; C.Y. Lu, et.al., Nature Physics **3**, 91(2007) .

[15] M. Hein, W. Dur, J. Eisert, R. Raussendorf, M. van den Nest and H. J. Briegel, eprint, arXiv: quant-ph/0602096.

[16] A. Cabello, A. J. Lopez-Tarrida, P. Moreno, and J. R. Portillo, eprint, arXiv: 0812.4625[quant-ph]

[17] X. Y. Chen, eprint, arXiv: 0906.5130[quant-ph]